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EQUILIBRIUM OF AN ELASTIC BODY PIERCED BY HORIZONTAL THIN ELASTIC BARS

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A simple mathematical model of a structure consisting of a three-dimensional body and rigid carrying bars is proposed. The estimated characteristics are the deflections of the bars, their reactions averaged over the sections, and the subsidence parameters of the body. The problem formulated on the basis of asymptotic analysis comprises the bending equations of the bars, the equations of equilibrium of the body, and a relation between the reactions and the deflections of the bars. In this problem, in addition to the moment of inertia, another cross-sectional characteristic, namely, the outer conformal radius, is involved. The method of solving the problem and the ways of its generalization are discussed.

Introduction. In this paper, a simple method based on an asymptotic analysis is proposed to determine the stress-strain state of a three-dimensional elastic body joined to bars. Rajapakse and Wang [1] treated the problem of a loaded elastic bar embedded in an elastic half-space. Nazarov [2] and Kozlov et al. [3] constructed an asymptotic theory of joint to bars whose ends are welded to the surface of a body. For the structure under consideration, only an asymptotic solution of the heat conduction problem is known [4, 5].

The stress-strain state of the structure and the asymptotic form primarily depend on the relations between the moduli of elasticity of the bars and the body. For example, if the carrier bars are pliable, the body can be assumed undeformable in a first-order approximation, and the structure can be analyzed stage by stage. At the first stage, the deflections of the bars are determined; at the second stage, the body subsidence is determined; at the third stage, the stress-strain state of the body in proximity to the bars is refined by the method of boundary layers (the solution of the problems for a half-space with a cylindrical cavity [6]). Argatov and Nazarov [4, 5] showed that the problem is not "split" if the bars are sufficiently rigid. This most complicated case is studied below.

Formulation of the Problem. We consider a structure (Fig. 1) consisting of a body Ω_ε and thin straight bars Q_ε^1 and Q_ε^2, where $\varepsilon > 0$ is a small parameter which characterizes the cross-sectional dimensions of the bars. We denote the density, the Young's modulus, and the Poisson ratio of the body and the bars by ρ, E, and ν and ρ_j, E_j, and ν_j ($j = 1, 2$), respectively. All the structural elements are made from homogeneous isotropic linear-elastic materials, the engagement between the bars and the body being ideal (no the slippage and delamination). The structure undergoes small deformations due to gravity, the surface loads are absent, and the ends of the bars are rigidly fixed.

We consider the problem of determining the forces transferred from Ω_ε to Q_ε^1 and Q_ε^2. It is assumed that the axes of the bars (the straight lines $x_1 = x_1^j$ and $x_2 = 0$) are parallel to the Ox_3 axis and lie in the horizontal plane $x_2 = 0$. Moreover, the cross section ω_ε^j of the bar Q_ε^j is bounded by an ellipse whose principal axes have lengths $2a_\varepsilon^j = 2\varepsilon A_j$ and $2b_\varepsilon^j = 2\varepsilon B_j$ and are parallel to the Ox_1 and Ox_2 axes.

Determination of the Displacements of the Bars. The bars Q_ε^1 and Q_ε^2 of length $2l_1$ and $2l_2$, respectively, are bent by the weight of the body. We denote the vertical deflection of the axis of the bar w^j

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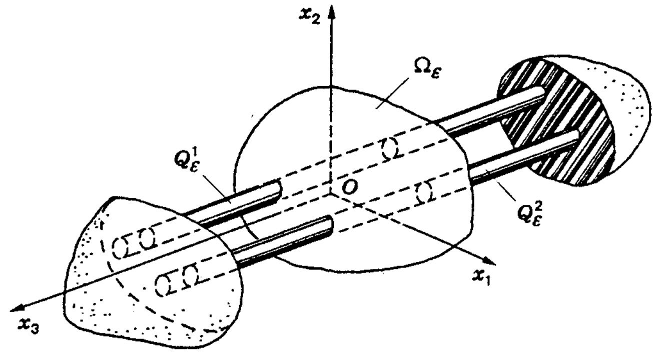

Fig. 1

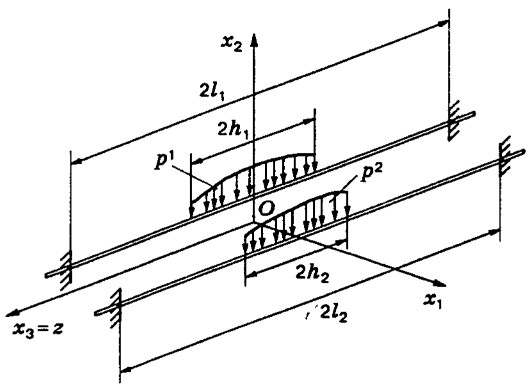

Fig. 2

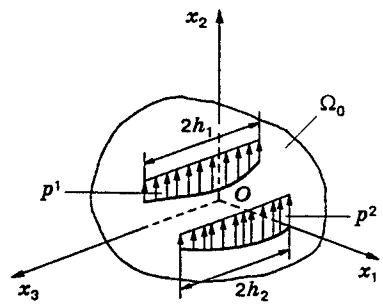

Fig. 3

by Q^j_ε. The moment of inertia of the cross section ω^j_ε about the Ox_1 axis is calculated by the formula

$$I^j_\varepsilon = \frac{\pi}{4} a^j_\varepsilon (b^j_\varepsilon)^3 = \varepsilon^4 \frac{\pi}{4} A_j B^3_j. \tag{1}$$

We assume that the bars and their parts which contact with the body are symmetric about the Ox_1x_2 plane.

Using the engineering theory of bending, we represent the action of the elastic body on the bars by distributed loads (Fig. 2). The load p^j which is unknown at the contact segment is introduced into the right-hand side of the differential equation for w^j:

$$E_j I^j_\varepsilon \frac{d^4 w^j}{dz^4}(z) = -q^j_\varepsilon - p^j(z), \qquad |z| < h_j; \tag{2}$$

$$E_j I^j_\varepsilon \frac{d^4 w^j}{dz^4}(z) = -q^j_\varepsilon, \qquad h_j \leqslant |z| < l_j. \tag{3}$$

Here $j = 1,\ 2$, $z = x_3$, and q^j_ε is the weight per unit length of the bar Q^j_ε:

$$q^j_\varepsilon = \pi a^j_\varepsilon b^j_\varepsilon \rho_j g = \varepsilon^2 \pi A_j B_j \rho_j g, \tag{4}$$

where g is the acceleration of gravity. The ends of the bars are assumed to be clamped:

$$w^j(\pm l_j) = 0, \qquad \frac{dw^j}{dz}(\pm l_j) = 0. \tag{5}$$

Provided the point loads are absent, the functions w^1 and w^2 are three times continuously differentiable.

Displacement Field Far from the Bars. Modeling the action of the bar Q^j_ε on the body by the loads p^j distributed along a segment of length $2h_j$ (Fig. 3), we arrive at the second fundamental problem of the theory of elasticity for the region Ω_0. Its solution is determined with an accuracy to a rigid displacement and the corresponding vector field can be represented in the form

$$\boldsymbol{v}(\boldsymbol{x}) = \alpha_2 \boldsymbol{e}_2 + \beta_1(x_2\boldsymbol{e}_3 - x_3\boldsymbol{e}_2) + \beta_3(x_1\boldsymbol{e}_2 - x_2\boldsymbol{e}_1) + \sum_{i=1}^{2}\int_{-h_i}^{h_i} p^i(s)\boldsymbol{G}^{(2)}[\boldsymbol{x};\boldsymbol{\xi}^i(s)]\,ds, \tag{6}$$

where α_2, β_1, and β_3 are the subsidence parameters of the body (the vertical displacement and the angles of rotation about the axes Ox_1 and Ox_3), and $\boldsymbol{G}^{(2)}[\boldsymbol{x};\boldsymbol{\xi}^i(s)]$ is the Green vector function for the region Ω_0 which corresponds to a unit point force applied at the point $\boldsymbol{\xi}^i(s) = (x_1^i, 0, s)$ and directed along the Ox_2 axis (see, e.g., [7, pp. 175–183]).

The loads p^1 and p^2 must balance statically the gravitational forces, i.e.,

$$\sum_{i=1}^{2}\int_{-h_i}^{h_i} p^i(s)\,ds + F_2 = 0,$$

$$-\sum_{i=1}^{2}\int_{-h_i}^{h_i} sp^i(s)\,ds - x_3^*F_2 = 0, \qquad \sum_{i=1}^{2}\int_{-h_i}^{h_i} x_1^i p^i(s)\,ds + x_1^*F_2 = 0. \tag{7}$$

Here $F_2 = -\rho|\Omega_0|g$, $|\Omega_0|$ is the volume of Ω_0, and x_1^* and x_3^* are the coordinates of the center of gravity of Ω_0, which differ slightly (by the order ε) from the coordinates of the center of gravity of Ω_ε.

The following decomposition is valid:

$$\boldsymbol{G}^{(2)}(\boldsymbol{x};\boldsymbol{\xi}) = \boldsymbol{T}^{(2)}(\boldsymbol{x}-\boldsymbol{\xi}) + \boldsymbol{g}^{(2)}(\boldsymbol{x};\boldsymbol{\xi}). \tag{8}$$

Here $\boldsymbol{T}^{(2)}$ is the Kelvin solution for an elastic half-space loaded with the unit force in the direction of Ox_2 axis

$$\boldsymbol{T}^{(2)} = \frac{1+\nu}{8\pi E(1+\nu)}\Big(\frac{x_1x_2}{|\boldsymbol{x}|^3}, (3-4\nu)\frac{1}{|\boldsymbol{x}|} + \frac{x_2^2}{|\boldsymbol{x}|^3}, \frac{x_2x_3}{|\boldsymbol{x}|^3}\Big). \tag{9}$$

The components of the vector $\boldsymbol{T}^{(2)}(\boldsymbol{x}-\boldsymbol{\xi})$ increase unboundedly as $\boldsymbol{x} \to \boldsymbol{\xi}$, while the components of the regular part $\boldsymbol{g}^{(2)}(\boldsymbol{x};\boldsymbol{\xi})$ of the Green vector function remain bounded. The field (6) also has logarithmic singularities; therefore, it approximates the actual displacement field in Ω_ε only at a distance from the bars. In other words, the field (6) is interpreted as a distant field or external representation of the solution (within the framework of the method of joined asymptotic expansions). The construction of the internal representations in the vicinity of the bars Q_ε^1 and Q_ε^2 and matching of these representations with the external representation are dealt with in the following three sections of the paper.

Displacement Field in the Vicinity of the Bars. We introduce the coordinates $(\boldsymbol{y}^j, z)$ referred to the bar Q_ε^j and the extended (fast) variables $\boldsymbol{\eta}^j$ in the planes perpendicular to its axis:

$$y_1^j = x_1 - x_1^j, \qquad y_2^j = x_2 - x_2^j, \qquad z = x_3; \tag{10}$$

$$\eta_1^j = \varepsilon^{-1}y_1^j, \qquad \eta_2^j = \varepsilon^{-1}y_2^j. \tag{11}$$

In the coordinates $(\boldsymbol{\eta}^j, z)$, the semi-axes of the elliptic sections of the bar are equal to A_j and B_j, i.e., they are independent of the parameter ε. Furthermore, the problem is formulated in an infinite region, since, in the plane $x_3 = z$, the external boundary Ω_ε is shifted at a distance $O(\varepsilon^{-1})$ due to the change $\boldsymbol{y}^j \mapsto \boldsymbol{\eta}^j$ and vanishes in the limiting case. The internal representation is described by the solution of the planar problem with the parameter $z \in (-h_j, h_j)$ (the three-dimensional boundary layer is ignored in the zones where the bars emanate from the body):

$$\boldsymbol{V}^j(\boldsymbol{\eta}^j, z) = w^j(z)\boldsymbol{e}_2 + p^j(z)[W_1^{(2)j}(\boldsymbol{\eta}^j)\boldsymbol{e}_1 + W_2^{(2)j}(\boldsymbol{\eta}^j)\boldsymbol{e}_2], \tag{12}$$

where $\boldsymbol{W}^{(2)j} = (W_1^{(2)j}, W_2^{(2)j})$ is the solution of the problem of deformation of an elastic plane by a unit force acting on an undeformed elliptic core

$$\omega_1^j = \{\boldsymbol{\eta}^j : (\eta_1^j/A_j)^2 + (\eta_2^j/B_j)^2 \leqslant 1\}.$$

The explicit form of $\boldsymbol{W}^{(2)j}$ is determined by known complex potentials [8, pp. 316–318]; the vector $\boldsymbol{W}^{(2)j}$ vanishes at $\partial\omega_1^j$ and admits the representation

$$\boldsymbol{W}^{(2)j}(\boldsymbol{\eta}^j) = \boldsymbol{S}^{(2)}(\boldsymbol{\eta}^j/R_j) + O(1) \quad \text{as} \quad |\boldsymbol{\eta}^j| \to \infty. \tag{13}$$

Here $\boldsymbol{S}^{(2)}$ is the solution of the problem of an elastic plane loaded by a unit point load directed along the axis of ordinates, and $R_j = (A_j + B_j)/2$ is the outer conformal radius of the ellipse with the semi-axes A_j and B_j (see, e.g., [9, pp. 18–20]). If $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$ are the dimensionless coordinates, we have

$$\boldsymbol{S}^{(2)}(\boldsymbol{\zeta}) = \frac{1+\nu}{4\pi E(1-\nu)}\Big(\frac{\zeta_1\zeta_2}{|\boldsymbol{\zeta}|^2}, -(3-4\nu)\ln|\boldsymbol{\zeta}| + \frac{\zeta_2^2}{|\boldsymbol{\zeta}|^2}\Big).$$

Owing to the term $w^j(z)\boldsymbol{e}_2$ and the vanishing of $W_i^{(2)j}$ at $\partial\omega_1^j$, the internal representation (12) agrees principally with the displacement field of the bar Q_ε^j, which is determined using the engineering theory of bending. At the same time, by virtue of (13), at a distance from the contact surface, the behavior of the vector (12) is characterized by the relation

$$\boldsymbol{V}^j(\boldsymbol{\eta}^j, z) = w^j(z)\boldsymbol{e}_2 + p^j(z)\boldsymbol{S}^{(2)}(\boldsymbol{\eta}^j/R_j) + O(1) \quad \text{as} \quad |\boldsymbol{\eta}^j| \to \infty. \tag{14}$$

The asymptotic terms in (14) must be joined with the asymptote of the external representation (6).

Behavior of the External Representation Near the Bars. We substitute (10) into (6) and take (8) and (9) into account. The resulting expression for $\gamma(x)$ includes, in particular, the integral

$$\int_{-h_i}^{h_i} \frac{p^j(s)\,ds}{|\boldsymbol{x} - \boldsymbol{\xi}^j(s)|} = \int_{-h_i}^{h_i} \frac{p^j(s)\,ds}{\sqrt{|\boldsymbol{y}^j|^2 + (z-s)^2}}. \tag{15}$$

We find its principal asymptotic term as $|\boldsymbol{y}^j| \to 0$. We represent the integral (15) in the form of the sum [10, 11]

$$\int_{-h_i}^{h_i} \frac{p^j(s)\,ds}{\sqrt{|\boldsymbol{y}^j|^2 + (z-s)^2}} = p^j(z)\int_{-h_i}^{h_i} \frac{ds}{\sqrt{|\boldsymbol{y}^j|^2 + (z-s)^2}} + \int_{-h_i}^{h_i} \frac{p^j(s) - p^j(z)}{\sqrt{|\boldsymbol{y}^j|^2 + (z-s)^2}}\,ds.$$

If the function p^j is smooth, the second integral is bounded as $|\boldsymbol{y}^j| \to 0$ and the first integral is expressed in terms of elementary functions and has the asymptote

$$\int_{-1}^{1} \frac{dt}{\sqrt{(|\boldsymbol{y}^j|/h_j)^2 + [(z/h_j) - t]^2}} = -2\ln\frac{|\boldsymbol{y}^j|}{h_j} + O(1) \quad \text{as} \quad |\boldsymbol{y}^j| \to 0.$$

The other integrals which diverge as $|\boldsymbol{y}^j| \to 0$ are transformed similarly. As a result, for the vector field (6), the asymptotic formula

$$\boldsymbol{v}(x_1^j + y_1^j, y_2^j, z) = (\alpha_2 - \beta_1 z + \beta_3 x_1^j)\boldsymbol{e}_2 + p^j(z)\boldsymbol{S}^{(2)}(\boldsymbol{y}^j/h_j) + O(1) \quad \text{as} \quad |\boldsymbol{y}^j| \to 0. \tag{16}$$

is valid.

Matching of the External and Internal Representations. Using the method of joined asymptotic expansions [12, 13], in relation (14), we pass from the fast $\boldsymbol{\eta}^j$ to the slow $\boldsymbol{y}^j = \varepsilon\boldsymbol{\eta}^j$ coordinates. In the overlap region where the asymptotic representations (6) and (12) are applicable, we find

$$\boldsymbol{V}^j(\varepsilon^{-1}\boldsymbol{y}^j, z) = w^j(z)\boldsymbol{e}_2 + p^j(z)\boldsymbol{S}^{(2)}(\boldsymbol{y}^j/(\varepsilon R_j)) + O(1), \quad \sqrt{\varepsilon}R_j \leqslant |\boldsymbol{y}^j| \leqslant 2\sqrt{\varepsilon}R_j, \quad \varepsilon \to 0. \tag{17}$$

We compare (16) and (17). In order that the expressions on the right-hand sides of these formulas coincide, it is necessary and sufficient that the equality

$$\alpha_2 - \beta_1 z + \beta_3 x_1^j + \frac{(1+\nu)(3-4\nu)}{4\pi E(1-\nu)} p^j(z) \ln \frac{h_j}{\varepsilon R_j} = w^j(z) \qquad (|z| < h_j, \quad j = 1, 2) \tag{18}$$

hold. Thus, the equations of the problem have been formulated.

Discussion of the Equations of the Problem. The asymptotic solution of the initial problem involves the unknown functions w^j and p^j ($j = 1$ and 2) and the parameters α_2, β_1, and β_3. These are found by solving a problem that comprises the differential bending equations of the bars (2) and (3), the boundary conditions (5), the equations of equilibrium of the body (7), and the governing functional relation (18).

Equation (18) involves the large (as $\varepsilon \to 0$) parameter $|\ln \varepsilon|$. Therefore, the parameters α_2, β_1, and β_3 and the functions w^j ($j = 1$) are of the order $|\ln \varepsilon|$. The terms on the right-hand side of Eq. (2) are of the same order if $\rho_j = \varepsilon^{-2}\rho_j^*$ and ρ and ρ_j^* are independent of ε [compare (4) with (7)]. Finally, relations (2), (3), (5), (7), and (18) form a unified problem only if the condition

$$E_j = \varepsilon^{-4} |\ln \varepsilon|^{-1} E_j^* \tag{19}$$

holds, E and E_j^* being of the same order [see (1), (2), and (18)]. If the bars are thin and the material is sufficiently rigid, the small parameter ε is not excluded from the problem completely and an additional asymptotic analysis cannot simplify the problem, i.e., it cannot separate it into several problems solved in succession for each of the structural elements as in the case of pliable bars. If the moduli of elasticity of the bars exceed the modulus of elasticity of the body to a greater extent as is required by condition (19), the bars can be assumed to be undeformable, i.e., one can set $w^j(z) = 0$ for $|z| < l_j$ and thus "split" the problem. First, from Eq. (18) with the zero right-hand side and from Eq. (7), the loads p^j and the subsidence are found. Then, after substitution of the calculated p^j into (2), the deflections w^j are refined. In addition to the moment of inertia I_ε^j, the problem involves another important geometrical cross-sectional characteristic of the bar Q_ε^j , namely, the outer conformal radius

$$r_\varepsilon^j = \varepsilon R_j = \left(a_\varepsilon^j + b_\varepsilon^j\right)/2. \tag{20}$$

The outer conformal radius can be taken as the mean characteristic dimension of the cross section (see, e.g., [14]).

Using (20), in the governing equation (18), we introduce the notation for the stiffness coefficient

$$k_\varepsilon^j = \frac{4\pi E(1-\nu)}{(1+\nu)(3-4\nu)} \frac{1}{\ln(h_j/r_\varepsilon^j)}.$$

The subscript ε in the notation I_ε^j, q_ε^j, r_ε^j, and k_ε^j can be omitted.

Solution of the Problem. From Eq. (18), we find

$$p^j(z) = k_j w^j(z) - k_j(\alpha_2 - \beta_1 z + \beta_3 x_1^j), \qquad |z| < h_j, \quad j = 1, 2. \tag{21}$$

We ignore the weight of the bars. Substituting (21) into (2), we obtain the equation of the type of the Winkler foundation equation (see, e.g., [15, p. 265]):

$$E_j I_j \frac{d^4 w^j}{dz^4}(z) + k_j w^j(z) = k_j(\alpha_2 - \beta_1 z + \beta_3 x_1^j) \qquad (|z| < h_j).$$

Its solution can be written in the form

$$w^j(z) = \alpha_2 - \beta_1 z + \beta_3 x_1^j + C_1^j \cosh(æz)\cos(æz) + C_2^j \cosh(æz)\sin(æz)$$
$$+ C_3^j \sinh(æz)\cos(æz) + C_4^j \sinh(æz)\sin(æz), \qquad |z| < h_j, \tag{22}$$

where $æ_j^4 = k_j/(4E_jI_j)$ (for simplicity, the subscript j of $æ_j$ is omitted). The relation

$$p^j(z) = C_1^j \cosh(æz)\cos(æz) + C_2^j \cosh(æz)\sin(æz) + C_3^j \sinh(æz)\cos(æz) + C_4^j \sinh(æz)\sin(æz). \tag{23}$$

follows from (21). Integrating Eq. (3) and bearing in mind condition (5), we have

$$w^j(z) = \begin{cases} D_1^j(l_j+z)^3 + D_2^j(l_j+z)^2, & -l_j \leqslant z \leqslant -h_j, \\ E_1^j(l_j-z)^3 + E_2^j(l_j-z)^2, & h_j \leqslant z \leqslant l_j. \end{cases} \tag{24}$$

The eight integration constants in (22)–(24) are determined from the continuity conditions for the function w^j and its derivatives of up to the third order, inclusively, for $z=-h_j$ and $z=h_j$. Generally, it is necessary to solve a system of eight linear algebraic equations with the right-hand sides which depend linearly on α_2, β_1, and β_3. By virtue of the principle of superposition, we have

$$C_i^j = c_{i1}^j\alpha_2 + c_{i2}^j\beta_1 + c_{i3}^j\beta_3, \tag{25}$$

where c_{i1}^j, c_{i2}^j, and c_{i3}^j depend on h_j, l_j, and $æ_j$. Substituting (23) into the equations of equilibrium (7), in accordance with (25), we obtain a system of linear algebraic equations for α_2, β_1, and β_3. After the subsidence parameters of the body are calculated, formula (23) gives the distributed loads, and formulas (22) and (24) give the deflections of the rods.

Conclusions. The constructed approximate solution is the more exact, the smaller the parameter ε. However, in comparison with other relations of the problem, the accuracy of the governing equation (18) and, consequently, the accuracy of the whole model is not high, since in the derivation of (18), we ignored terms of the order $|\ln\varepsilon|^{-1}$ compared to unity ($\varepsilon \ll 1$).

We shall mention the main drawbacks of the proposed asymptotic model and determine some ways of its generalization.

1. The shape of the body Ω_0 is not taken into account. To introduce the corresponding corrections into the governing equation (18), it is necessary to refine the behavior of the external asymptotic representation in the neighborhood of the bars [formula (16)]. For this purpose, information on the structure of the regular part $\boldsymbol{g}(\boldsymbol{x};\boldsymbol{\xi})$ of the Green vector function (8) that depends on Ω_0 is required.

2. The stress-strain state of the body in the zones where the bars emanate is three-dimensional and, obviously, it cannot be described by plane boundary layers. In these zones, it is necessary to construct three-dimensional boundary layers, i.e., the solutions of special problems of the theory of elasticity for a half-space with an undeformable cylindrical core; however, neither the explicit solution of this problem nor the theorems on its solvability are known.

3. The governing equation does not take into account the cross-sectional form of the bars, and only the characteristic dimension of the cross section enters (18). To refine the model, one should calculate the subsequent term in the asymptotic relation (13).

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